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Strength Distribution of Repeatedly Broken Chains

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We determine the probability distribution of the breaking strength for chains of N links, which have been produced by repeatedly breaking a very long chain.

KEY WORDS: fragmentation, chain breaking

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1. INTRODUCTION

Consider a chain assembled from N random links with independent, identically distributed breaking strengths x, which have probability density $\rho(x)$. It is easy to calculate the probability density $\rho(x|N)$ for the strength of the chain being less than x. In some contexts, however, the relevant question is a different one: what are the strengths of chain segments of length N which are obtained by repeatedly breaking a very long chain? Such chain segments are expected to be stronger than their randomly assembled counterparts, because they have been produced by a process which has eliminated the weakest links. Here we calculate the distribution of the strength of a chain segment of length N which has been produced by repeated breaking a very long chain, of length $\mathcal{N}$, say. Specifically, we break the chain of length $\mathcal{N}$ at its weakest link, then break each fragment at its weakest link and continue the process. We collect the links of length N produced by this process, and determine the probability density of their strengths, $\rho^*(x|N)$. In this paper we obtain formulae which are precise asymptotic results for the limit where $N \gg 1$.

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Calculating this distribution involves the use of prior information: we know that the chain segment of length N and breaking strength x was produced by breaking a longer and weaker chain, of length N_0 and breaking strength x_0. What makes the problem difficult is that the prior information, in the form of the values of N_0 and x_0, is itself uncertain. It is hard to solve this problem for the case of a chain which has previously been broken only once. In the case of a chain which has been broken many times, the distributions of x_0 and N_0 themselves depend on previous breakages: the problem appears to be very difficult when the chain has previously been broken many times.

Since we consider the case where the chain is very long, we anticipate that there is an asymptotic form for the strength distribution which is independent of the initial length $\mathcal{N}$. We use a self-consistent calculation and obtain $\rho^*(x|N)$ in closed form in terms of the cumulative strength distribution of an individual link, $P(X)$, and the corresponding probability density $\rho(x) = \mathrm{d}P(x)/\mathrm{d}x$:

$$\rho^*(x|N) \sim \frac{1}{2} N^3 \rho(x)[P(x)]^2 \exp[-NP(x)]. \tag{1}$$

(We use $A(N) \sim B(N)$ to mean that $A(N)$ and $B(N)$ are asymptotically equal in the limit as $N \to \infty$.) For comparison, the corresponding probability density for the strength of a randomly assembled chain is

$$\rho(x|N) \sim N\rho(x) \exp[-NP(x)]. \tag{2}$$

Figure 1 shows the comparison between Eq. (1) and a histogram of data from a numerical simulation of the chain-breaking process, for a particular choice of link strength distribution $\rho(x)$.

One of our motivations for studying the chain-breaking process was our observation (in numerical experiments) that this process has connections with Mott's variable-range hopping problem.[(1)] Specifically, the distribution of lengths of current-carrying bonds in the one-dimensional case of Mott's variable-range-hopping model is closely related to the distribution of lengths of unbreakable segments in breaking a random chain.[(2)] The model we solve below is a version of the chain-breaking model for variable-range hopping which is simple enough to be exactly solvable.

The process we describe here is also of interest because it models fragmentation processes in which the distribution of *strengths* of fragments (rather than the distribution of fragment *sizes*) is the principal concern. A number of authors have previously discussed the distribution of fragment sizes.[(3–5)] Other works have discussed fragment-size distributions in more general random fragmentation processes, for example so-called 'stick-breaking processes',[(6)] and are of interest in describing genetic variation in biological populations, see for example.[(6–8)]

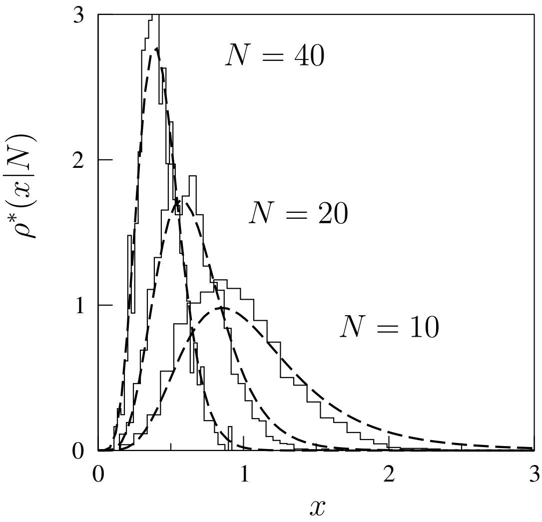

Fig. 1. Shows $\rho^*(x|N)$ for three different values of N, namely 10, 20 and 40, for $\rho(x) = x\,\exp(-x)$: computer simulations (histogram) are compared with Eq. (1), which takes the form $\rho^*(x|N) = (N^3/2)\,x\,e^{-x}[1-(1+x)e^{-x}]^2 e^{-N(1-(1+x)e^{-x})}$.

We are not aware of any earlier investigations of the strength distribution of fragments. The chain-breaking process is of interest because it is a fundamental model for such processes, which has an explicit solution for the distribution $\rho^*(x|N)$.

In Sec. 2 we consider the distribution of the sizes of fragments of a repeatedly broken chain, and Sec. 3 describes elementary results for randomly assembled chains. In Sec. 4 these expressions are used to obtain a recursion for a sequence of distributions which are related to $\rho^*(x|N)$. Finally Sec. 5 shows how the limiting distribution of this sequence is obtained self-consistently, and used to derive Eq. (1) above.

2. DISTRIBUTION OF FRAGMENT SIZES

First we discuss the distribution of sizes of chain fragments. After i steps of splitting the chain, we have 2^i fragments. Let $W_i(N)$ be the number of segments of lengths N at step i, and let us consider the case where these numbers are so large that it is sufficient to calculate expectation values and ignore the statistical fluctuations of $W_i(N)$ (this assumption is valid in the limit as $\mathcal{N} \to \infty$). Using the fact that the position of the weakest link has equal probability to be at any site, these numbers satisfy a recursion relation:

$$W_{i+1}(N) = \sum_{M=N+1}^{\infty} \frac{2}{M-1} W_i(M). \tag{3}$$

Rather than following this iteration for a single chain being broken, it is easier to consider a steady state $W(N)$, with destruction of one additional chain being initiated at each step. Thus we seek to solve

$$W(N) = \sum_{M=N+1}^{\infty} \frac{2}{M-1} W(M). \tag{4}$$

We find $W(N) \sim C/N^2$ for $N \to \infty$, for some constant C.

3. STRENGTH OF A RANDOM CHAIN SEGMENT

By way of preparation we discuss elementary results on the distribution of strengths for a chain with completely random links. Let $P(x|N)$ be the probability that a chain of length N breaks at a tension which is less than x. For $N = 1$, we have $P(x|1) = P(x)$. The probability that a chain of N links is unbreakable at tension x is $1 - P(x|N) = [1 - P(x)]^N$. In the limit where $N \gg 1$, the power may be approximated by exponentiation: we find $P(x|N) \sim 1 - \exp[-NP(x)]$ (which gives (2) upon differentiation).

We also require the conditional probability $P_c(x|x_0, N)$ that a chain of length N breaks at tension x if we know that it is definitely not broken by a tension x_0. In this case we know that the probability of the strength of an individual element being less than x is $[P(x) - P(x_0)]/[1 - P(x_0)]$, so that

$$P_c(x|x_0, N) = 1 - \left[\frac{P(x) - P(x_0)}{1 - P(x_0)}\right]^N \tag{5}$$

provided that $x > x_0$ (and zero otherwise). When $N \gg 1$, this may be approximated by

$$P_c(x|x_0, N) \sim \big[1 - \exp\big(-N[P(x) - P(x_0)]\big)\big]\Theta(x - x_0) \tag{6}$$

where $\Theta(x)$ is the unit increasing step function (Heaviside function).

4. A RECURSION RELATION FOR PROBABILITIES OF SUCCESSIVE BREAKING TENSIONS

Our objective is to obtain the probability density $\rho^*(x|N)$ that a chain of length N, which has been produced by repeatedly breaking a very long chain, has breaking strength x.

We determine this distribution of strengths by calculating a related distribution. Consider the subdivision of the chain, starting from a very long chain of length $\mathcal{N}$. When this is repeatedly split, for convenience we always discuss the leftmost segment. At the ith stage of subdivision, this segment having length N_i is produced by breaking a link of strength x_i. Let the probability density for the strength of the link that was broken be $\rho_i(x_i|N_i)$. The corresponding probability for the ith split to occur at a tension less than x_i to produce a segment of length N_i is $P_i(x_i|N_i)$. We shall obtain a recursion formula which expresses $\rho_{i+1}(x_{i+1}|N_{i+1})$ in terms of this distribution. The chain fragment at stage $i+1$ is produced by breaking a segment of length N_i, in which all of the links are known to be stronger than x_i. Both of these items of prior information (x_i and N_i) have uncertain values.

Consider first the distribution of values of N_i, for a given value of N_{i+1}. We have already seen (Eq. (4)) that repeated random sub-division of an interval produces a steady-state distribution of lengths $W(N) \sim 1/N^2$. Subdivision of an interval of length $N_i > N_{i+1}$ produces an interval of length N_{i+1} with probability $1/N_i$. The probability distribution for N_i is therefore proportional to N_i^{-3} for $N_i > N_{i+1}$ (and zero otherwise). For $N_{i+1} \gg 1$, the normalised distribution of N_i is therefore

$$P_{\mathrm{n}}(N_i|N_{i+1}) \sim \begin{cases} \dfrac{2N_{i+1}^2}{N_i^3} & N_{i+1} < N_i \\ 0 & N_{i+1} \geq N_i \end{cases} . \tag{7}$$

The probability $P_{i+1}(x_{i+1}|N_{i+1})$ is obtained from $P_{\mathrm{c}}(x_{i+1}|x_i, N_i)$ by averaging over the probability densities of both N_i and x_i. (Note that the latter probability density is related to the unknown function that we wish to calculate.):

$$P_{i+1}(x_{i+1}|N_{i+1}) = \sum_{N_i=N_{i+1}+1}^{\infty} P_{\mathrm{n}}(N_i|N_{i+1}) \int_0^{x_{i+1}} \mathrm{d}x_i \; P_{\mathrm{c}}(x_{i+1}|x_i, N_i)\rho_i(x_i|N_i). \tag{8}$$

Differentiating (8) with respect to x_{i+1} and substituting the known expressions for $P_{\mathrm{n}}(N_i|N_{i+1})$ and $P_{\mathrm{c}}(x_1|x_0, N_0)$ we obtain

$$\rho_{i+1}(x_{i+1}|N_{i+1}) \sim \rho(x_{i+1}) \sum_{N_i=N_{i+1}+1}^{\infty} \frac{2N_{i+1}^2}{N_i^2} \times \int_0^{x_{i+1}} \mathrm{d}x_i \; \exp(-N_i[P(x_{i+1}) - P(x_i)])\rho_i(x_i|N_i). \tag{9}$$

5. SELF-CONSISTENT SOLUTION

We assume that the distribution $\rho_i(x|N)$ becomes independent of the generation index i when we consider the subdivision of a very long chain. This leads to a 'self-consistency' condition for the probability density $\rho_i(x|N) = \mathrm{d}P_i(x|N)/\mathrm{d}x$. Approximating the sum in Eq. (9) by an integral, and replacing ρ_i and ρ_{i+1} by the asymptotic, self-consistent function ρ_∞ yields

$$\rho_\infty(x|N) \sim 2N^2\rho(x)\int_N^\infty \mathrm{d}N_0\,\frac{1}{N_0^2}\int_0^x \mathrm{d}x_0 \\ \times\ \exp(-N_0[P(x)-P(x_0)])\rho_\infty(x_0|N_0). \tag{10}$$

We assume that the corresponding probability $P_\infty(x|N)$ is of the form $P_\infty(x|N) = F(NP(x))$ for some function F (which increases monotonically from $F(0) = 0$ to $F(\infty) = 1$). Writing $w = P(x)$, the derivative $f = F'$ of F satisfies

$$f(Nw) = 2N\int_N^\infty \mathrm{d}N_0\,\frac{1}{N_0}\int_0^w \mathrm{d}w_0\ \exp[-N_0(w-w_0)]f(N_0w_0). \tag{11}$$

We express this integral equation in terms of $g(X) = \exp(X)f(X)$, and differentiate. We find that the primitive of $g(X)$, namely $G(X)$, satisfies

$$G''(X) = \frac{X+1}{X}G'(X) - \frac{2}{X}G(X). \tag{12}$$

The solution of this equation is of the form

$$G(X) = AX^2 + B[\exp(-X)(1+X) - X^2\mathrm{Ei}(X)] \tag{13}$$

where A, B are constants and $\mathrm{Ei}(X) = \mathcal{P}\int_{-\infty}^X \mathrm{d}z\,\mathrm{e}^z/z$ is the exponential integral. The requirement that $G(0) = 0$ implies that $B = 0$, so that the normalised function $f(x)$ is $f(x) = x\exp(-x)$. Thus we find the probability density for a chain segment of length N being formed by breaking a link of strength x in the form

$$\rho_\infty(x|N) = N^2\rho(x)P(x)\exp[-NP(x)]. \tag{14}$$

Finally, we are in the position to obtain the desired result, the probability density for a segment of length N having a weakest link of strength x. This is obtained from the distribution $\rho_\infty(x|N)$ as follows

$$\rho^*(x|N) = \int_0^x \mathrm{d}x_0\,\frac{\mathrm{d}}{\mathrm{d}x}P_\mathrm{c}(x|x_0,N)\rho_\infty(x_0|N) \\ = N\rho(x)\int_0^x \mathrm{d}x_0\ \exp\big(-N[P(x)-P(x_0)]\big)\rho_\infty(x_0|N). \tag{15}$$

In the limit of large N we find the simple, asymptotically exact result (1).

6. DISCUSSION

We conclude by briefly discussing the behaviour of our solution, equation (1), in different circumstances. In the case where $\rho(x)$ has a finite limit for very weak chains, $\rho_0 = \rho(0)$, the strength distribution (1) has a simple form when $N \to \infty$:

$$\rho^*(x|N) \sim \frac{1}{2} N^3 \rho_0^3 x^2 \exp(-N\rho_0 x). \tag{16}$$

This a universal function of the scaling variable $x\rho_0 N$ (note that the example shown in Fig. 1 is not of this scaling form).

Another case of interest is when the distribution of chain strengths is sharply peaked. We write

$$P(x) = \exp[-F(x)] \tag{17}$$

where $F(x)$ increases very rapidly as x approaches zero from above. We approximate each of Eqs. (2) and (1) by expanding about the point where (respectively) $\rho(x|N)$ or $\rho^*(x|N)$ is maximal. Let $x_{\max}$ be the position of the maximum of the strength distribution for the un-conditioned chain of length N, and $x^*_{\max}$ the position of the maximum for the chain produced by repeated breaking of longer fragments. We find:

$$F(x_{\max}) = \ln(N)\,, \quad F(x^*_{\max}) = \ln(N/3) \tag{18}$$

provided that N is not so large that these equations have no solution for positive values of $x_{\max}$, $x^*_{\max}$. These relations imply that in the large-N limit, conditioning the chains increases their expected strength.

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